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An Analytical Solution of Temperature Response in Multilayered Materials for Transient Methods

N. Araki,[1] A. Makino,[1] T. Ishiguro,[1] and J. Mihara[1]

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Transient methods, such as those with pulse- or stepwise heating, have often been used to measure thermal diffusivities of various materials including layered materials. The objective of the present study is to derive an analytical solution of the temperature rise in a multilayered material, the front surface of which is subjected to pulse- or stepwise heating. The Laplace transformation has been used to obtain the analytical solution. This solution will enable us to establish the appropriate measurement method for thermophysical properties of the multilayered material. It is also shown that the present solution can be extended to functionally gradient materials (FGM), in which thermophysical properties as well as compositions change continuously.

KEY WORDS: functionally gradient material; multilayered material; pulse-wise heating method; stepwise heating method; thermal diffusivity.

1. INTRODUCTION

Wide attention has been given to multilayered materials as electronic materials and materials resistant to wear, corrosion, and heat; they are anticipated to improve the specific nature of the conventional materials with homogeneity in composition, structure, and texture. The renovation to improve the homogeneity is expected to play an important role in technolgy because recent progress in technology requires advanced materials which will have the useful, multifunctional characteristics generated by heterogeniety in composition, structure, texture, etc. Among these advanced materials, functionally gradient materials (FGM), which are composed of different material components such as ceramics and

[1] Department of Energy and Mechanical Engineering, Shizuoka University, Hamamatsu 432, Japan.

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metals with continuous profiles in composition, structure, texture, mechanical strength, and thermophysical properties, have attracted special interest as advanced heat-shielding structural materials in future space applications.

In order for the FGM to qualify as advanced heat-shielding structural materials, the thermophysical properties as well as the mechanical properties are required to be evaluated properly. In addition, transient methods such as those with pulse- or stepwise heating are necessary for these evaluations because of their simplicity and usefulness at high temperatures. As a result, it becomes essential to use transient methods to evaluate the thermophysical properties of the multilayered materials, which become the FGM when the individual layer thickness is infinitesimal. However, application of the transient methods for the multilayered materials need great care because the thermal diffusivity obtained from the temperature response is apparent and different from that of averaged property related to the thermal resistance. Therefore, before making measurements by the transient methods, it is essential to investigate and confirm the measurement principles in the application of the transient methods to the multilayered materials.

The analytical solution of the temperature rise in the multilayered material, which is subjected to transient heating, is very limited. When the front surface is subjected to pulsewise heating, analytical solutions of the temperature rises are developed by Parker et al. [1] for a single-layered material and by Lee [2] for two- and three-layered materials. For the stepwise heating method, the analytical solution is developed by Araki and Natsui [3] for a three-layered material. However, no analytical solution for the multilayered material is reported. Therefore, it is required to develop the general analytical solution for arbitrary multilayered material in order to establish a measurement technique for the thermophysical properties and to evaluate the measured results.

The objectives and contributions of the present study are the following. First, a general analytical solution of the temperature rise in the multilayered material, which is subjected to the transient heating, is developed. This solution is easily extended to the temperature rise for the functionally gradient material (FGM) if the thickness of each layer is considered to be infinitesimal. It is shown that the temperature response for the FGM coincides with that for the single-layered material when one uses the Fourier number based on the apparent thermal diffusivity and the thickness of the sample material. The relation between the apparent thermal diffusivity obtained from the temperature response and the mean thermal diffusivity related to the thermal resistance is also given. Second, numerical calculations are performed in order to clarify the difference between the temperature rise due to the number of layers in the sample material when the thermophysical properties of the first and final layers are given. Also, the temperature response inside the multilayered material is calculated.

2. FORMULATION

2.1. Assumptions, Governing Equation, and Boundary Conditions

A multilayered material, the front surface of which is subjected to a pulsewise or stepwise radiation heat input, is considered and the subsequent temperature transient is investigated, by solving the heat diffusion equation with the appropriate boundary conditions. The following assumptions are made:

(1) one-dimensional heat flow,
(2) no heat loss from the sample surface,
(3) no thermal contact resistance between layers,
(4) heat input uniformly absorbed on the front surface,
(5) homogeneous layers, and
(6) constant thermophysical properties of each layer.

The schematic diagram of the geometry of the multilayered material is shown in Fig. 1.

The heat diffusion equation for each layer is mathematically described as

$$\frac{\partial \theta_i(z, t)}{\partial t} = a_i \frac{\partial^2 \theta_i(z, t)}{\partial z^2} \qquad (i = 1, 2, \ldots, n) \tag{1}$$

The boundary conditions are

$$-\lambda_1 \frac{\partial \theta_1(z_1, t)}{\partial z} = W(t) \tag{2}$$

$$\theta_{i-1}(z_i, t) = \theta_i(z_i, t) \qquad (i = 2, \ldots, n) \tag{3}$$

$$\lambda_{i-1} \frac{\partial \theta_{i-1}(z_i, t)}{\partial z} = \lambda_i \frac{\partial \theta_i(z_i, t)}{\partial z} \qquad (i = 2, \ldots, n) \tag{4}$$

$$-\lambda_n \frac{\partial \theta_n(0, t)}{\partial z} = 0 \tag{5}$$

and the initial condition is

$$\theta_i(z, 0) = 0 \qquad (i = 1, 2, \ldots, n) \tag{6}$$

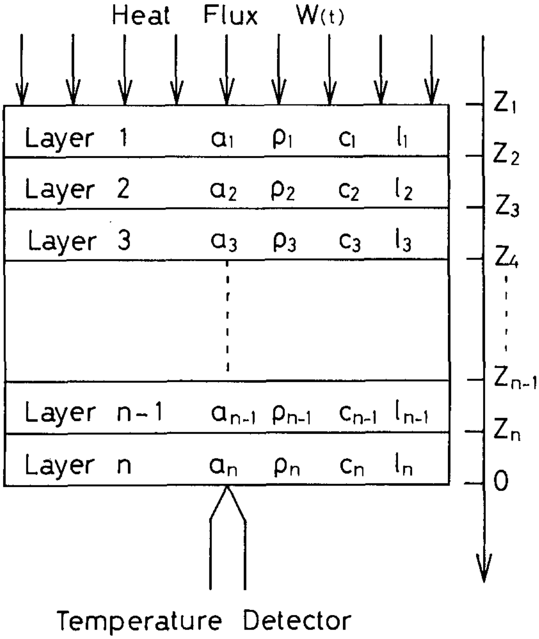

Fig. 1. Schematic diagram of the multilayered material.

where

$$l_i = z_{i+1} - z_i \tag{7}$$

and a is the thermal diffusivity, λ the thermal conductivity ($=\rho c a$), ρ the density, c the specific heat capacity, l the thickness, θ the temperature, z the distance from the rear surface, and t the time.

2.2. Solution by the Laplace Transformation

If we apply the Laplace transformation, we have the following solution from the subsidary equation:

$$\Theta_i(z, s) = A_i \sinh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z\right) + B_i \cosh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z\right) \quad (i = 1, 2, \ldots, n) \tag{8}$$

The constants A_i and B_i are determined with the following boundary conditions, treated in the same way:

$$-\lambda_1 \frac{d\Theta_1(z_1, s)}{dz} = W(s) \tag{9}$$

$$\Theta_{i-1}(z_i, s) = \Theta_i(z_i, s) \qquad (i = 2, \ldots, n) \tag{10}$$

$$\lambda_{i-1} \frac{d\Theta_{i-1}(z_i, s)}{dz} = \lambda_i \frac{d\Theta_i(z_i, s)}{dz} \qquad (i = 2, \ldots, n) \tag{11}$$

$$-\lambda_n \frac{d\Theta_n(0, s)}{dz} = 0 \tag{12}$$

Then we have

$$-\Lambda_1 \sqrt{s} \left\{ A_1 \cosh\left(\frac{\sqrt{s}}{\sqrt{a_1}} z_1\right) + B_1 \sinh\left(\frac{\sqrt{s}}{\sqrt{a_1}} z_1\right) \right\} = W(s) \tag{13}$$

$$\begin{aligned} A_{i-1} \sinh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) + B_{i-1} \cosh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) \\ = A_i \sinh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) + B_i \cosh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) \end{aligned} \tag{14}$$

$$\begin{aligned} A_{i-1} \cosh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) + B_{i-1} \sinh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) \\ = \Lambda_{i/i-1} \left\{ A_i \cosh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) + B_i \sinh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) \right\} \end{aligned} \tag{15}$$

$$A_n = 0 \tag{16}$$

with

$$\Lambda_i = \lambda_i / \sqrt{a_i}, \qquad \Lambda_{i/i-1} = \Lambda_i / \Lambda_{i-1} \tag{17}$$

Here Λ_i is called the heat-penetration coefficient, which depends only on the thermophysical properties and is independent of the thickness of the layer. If we put

$$\mathbf{b}_i = \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \quad \mathbf{C}_{i-1} = \begin{pmatrix} \cosh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) & \sinh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) \\ \sinh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) & \cosh\left(\frac{\sqrt{s}}{\sqrt{a_{i-1}}} z_i\right) \end{pmatrix},$$

$$\mathbf{D}_i = \Lambda_{i/i-1} \begin{pmatrix} \cosh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) & \sinh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) \\ \Lambda_{i-1/i} \sinh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) & \Lambda_{i-1/i} \cosh\left(\frac{\sqrt{s}}{\sqrt{a_i}} z_i\right) \end{pmatrix} \tag{18}$$

Equations (13) to (16) can be expressed as follows:

$$-\left[\frac{W(s)}{A_1\sqrt{s}}\right]=\left(\cosh\left(\frac{\sqrt{s}}{\sqrt{a_1}}z_1\right)\quad \sinh\left(\frac{\sqrt{s}}{\sqrt{a_1}}z_1\right)\right)\mathbf{b}_1 \tag{19}$$

$$\mathbf{C}_{i-1}\mathbf{b}_{i-1}=\mathbf{D}_i\mathbf{b}_i \tag{20}$$

$$\mathbf{b}_n=B_n\begin{pmatrix}0\\1\end{pmatrix} \tag{21}$$

Then from Eq. (20) we have

$$\mathbf{b}_1=\mathbf{C}_1^{-1}\mathbf{D}_2\mathbf{C}_2^{-1}\mathbf{D}_3\cdots\mathbf{C}_{i-1}^{-1}\mathbf{D}_i\mathbf{C}_i^{-1}\mathbf{D}_{i+1}\cdots\mathbf{C}_{n-2}^{-1}\mathbf{D}_{n-1}\mathbf{C}_{n-1}^{-1}\mathbf{D}_n\mathbf{b}_n \tag{22}$$

If we define that

$$\mathbf{E}_1\equiv\begin{pmatrix}\cosh(\sqrt{s}\,\eta_n\eta_{1/n}) & -\sinh(\sqrt{s}\,\eta_n\eta_{1/n})\\ -\sinh(\sqrt{s}\,\eta_n\eta_{1/n}) & \cosh(\sqrt{s}\,\eta_n\eta_{1/n})\end{pmatrix} \tag{23}$$

$$\begin{aligned}\mathbf{E}_i&\equiv\mathbf{D}_i\mathbf{C}_i^{-1}\\ &=A_{i/i-1}\begin{pmatrix}\cosh(\sqrt{s}\,\eta_n\eta_{i/n}) & -\sinh(\sqrt{s}\,\eta_n\eta_{i/n})\\ -A_{i-1/i}\sinh(\sqrt{s}\,\eta_n\eta_{i/n}) & A_{i-1/i}\cosh(\sqrt{s}\,\eta_n\eta_{i/n})\end{pmatrix}\end{aligned} \tag{24}$$

$$\mathbf{E}_n\equiv\mathbf{D}_n=A_{n/n-1}\begin{pmatrix}\cosh(\sqrt{s}\,\eta_n) & -\sinh(\sqrt{s}\,\eta_n)\\ -A_{n-1/n}\sinh(\sqrt{s}\,\eta_n) & A_{n-1/n}\cosh(\sqrt{s}\,\eta_n)\end{pmatrix} \tag{25}$$

with

$$\eta_i=l_i/\sqrt{a_i},\qquad \eta_{i/n}=\eta_i/\eta_n\qquad (i=1,\ldots,n) \tag{26}$$

Equation (22) becomes

$$\mathbf{b}_1=B_n\mathbf{C}_1^{-1}\mathbf{E}_1^{-1}\mathbf{x} \tag{27}$$

where

$$\mathbf{x}=\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_{i-1}\mathbf{E}_i\cdots\mathbf{E}_{n-1}\mathbf{E}_n\begin{pmatrix}0\\1\end{pmatrix} \tag{28}$$

$$\mathbf{C}_1^{-1}\mathbf{E}_1^{-1}=\begin{pmatrix}\cosh\left(\frac{\sqrt{s}}{\sqrt{a_1}}z_1\right) & -\sinh\left(\frac{\sqrt{s}}{\sqrt{a_1}}z_1\right)\\ -\sinh\left(\frac{\sqrt{s}}{\sqrt{a_1}}z_1\right) & \cosh\left(\frac{\sqrt{s}}{\sqrt{a_1}}z_1\right)\end{pmatrix} \tag{29}$$

Here η_i is called the thermal diffusion time. Substituting Eq. (27) into Eq. (19), we obtain

$$-\left[\frac{W(s)}{B_n A_1 \sqrt{s}}\right] = U(s) \tag{30}$$

where

$$U(s) = (1 \quad 0)\, \mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_{i-1} \mathbf{E}_i \cdots \mathbf{E}_{n-1} \mathbf{E}_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{31}$$

Then the temperature response at the rear surface can be expressed as

$$\Theta_n(0, s) = B_n = -\left[\frac{W(s)}{A_1 \sqrt{s}\, U(s)}\right] \tag{32}$$

2.3. Determination of the Characteristic Equation $U(s)$

In order to obtain the temperature response, we have to determine $U(s)$ of Eq. (31). Although $U(s)$ is known for up to three-layered material, it is not reported for multilayered material. In order to obtain $U(s)$ for the general case, let us first confirm $U(s)$ for up to $n = 3$. For the single-layered material ($n = 1$), $U(s)$ is obtained as follows from Eq. (23):

$$U(s) = -\sinh(\sqrt{s}\, \eta_1) \tag{33}$$

When $n = 2$ for a two-layered material, $U(s)$ is obtained as follows:

$$\begin{aligned} U(s) = -\left(\frac{A_{2/1}}{2}\right) \{&(A_{1/2} + 1) \sinh[\sqrt{s}\, \eta_2(\eta_{1/2} + 1)] \\ &+ (A_{1/2} - 1) \sinh[\sqrt{s}\, \eta_2(\eta_{1/2} - 1)]\} \end{aligned} \tag{34}$$

because of

$$\mathbf{E}_1 \mathbf{E}_2 = \left(\frac{A_{2/1}}{2}\right) \begin{pmatrix} E_{11}^{(2)} & E_{12}^{(2)} \\ E_{21}^{(2)} & E_{22}^{(2)} \end{pmatrix} \tag{35}$$

with

$$\begin{aligned} E_{11}^{(2)} = &(A_{1/2} + 1) \cosh\{\sqrt{s}\, \eta_2(\eta_{1/2} + 1)\} \\ &- (A_{1/2} - 1) \cosh\{\sqrt{s}\, \eta_2(\eta_{1/2} - 1)\} \end{aligned} \tag{36}$$

$$\begin{aligned} E_{12}^{(2)} = &-(A_{1/2} + 1) \sinh\{\sqrt{s}\, \eta_2(\eta_{1/2} + 1)\} \\ &- (A_{1/2} - 1) \sinh\{\sqrt{s}\, \eta_2(\eta_{1/2} - 1)\} \end{aligned} \tag{37}$$

$$\begin{aligned} E_{21}^{(2)} = &-(A_{1/2} + 1) \sinh\{\sqrt{s}\, \eta_2(\eta_{1/2} + 1)\} \\ &+ (A_{1/2} - 1) \sinh\{\sqrt{s}\, \eta_2(\eta_{1/2} - 1)\} \end{aligned} \tag{38}$$

$$E_{22}^{(2)} = (A_{1/2}+1)\cosh\{\sqrt{s}\,\eta_2(\eta_{1/2}+1)\} + (A_{1/2}-1)\cosh\{\sqrt{s}\,\eta_2(\eta_{1/2}-1)\} \tag{39}$$

When $n = 3$ for a three-layered material, $U(s)$ is obtained as follows:

$$\begin{aligned} U(s) = -\left(\frac{A_{3/1}}{2^2}\right)[&(A_{1/2}+1)(A_{2/3}+1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}+1)\} \\ &+ (A_{1/2}+1)(A_{2/3}-1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}-1)\} \\ &+ (A_{1/2}-1)(A_{2/3}-1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}+1)\} \\ &+ (A_{1/2}-1)(A_{2/3}+1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}-1)\}] \end{aligned} \tag{40}$$

because of

$$\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3 = \left(\frac{A_{3/1}}{2^2}\right)\begin{pmatrix} E_{11}^{(3)} & E_{12}^{(3)} \\ E_{21}^{(3)} & E_{22}^{(3)} \end{pmatrix} \tag{41}$$

with

$$\begin{aligned} E_{11}^{(3)} = &(A_{1/2}+1)(A_{2/3}+1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}+1)\} \\ &- (A_{1/2}+1)(A_{2/3}-1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}-1)\} \\ &+ (A_{1/2}-1)(A_{2/3}-1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}+1)\} \\ &- (A_{1/2}-1)(A_{2/3}+1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}-1)\} \end{aligned} \tag{42}$$

$$\begin{aligned} E_{12}^{(3)} = &-(A_{1/2}+1)(A_{2/3}+1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}+1)\} \\ &- (A_{1/2}+1)(A_{2/3}-1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}-1)\} \\ &- (A_{1/2}-1)(A_{2/3}-1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}+1)\} \\ &- (A_{1/2}-1)(A_{2/3}+1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}-1)\} \end{aligned} \tag{43}$$

$$\begin{aligned} E_{21}^{(3)} = &-(A_{1/2}+1)(A_{2/3}+1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}+1)\} \\ &+ (A_{1/2}+1)(A_{2/3}-1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}-1)\} \\ &- (A_{1/2}-1)(A_{2/3}-1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}+1)\} \\ &+ (A_{1/2}-1)(A_{2/3}+1)\sinh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}-1)\} \end{aligned} \tag{44}$$

$$\begin{aligned} E_{22}^{(3)} = &(A_{1/2}+1)(A_{2/3}+1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}+1)\} \\ &+ (A_{1/2}+1)(A_{2/3}-1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}+\eta_{2/3}-1)\} \\ &+ (A_{1/2}-1)(A_{2/3}-1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}+1)\} \\ &+ (A_{1/2}-1)(A_{2/3}+1)\cosh\{\sqrt{s}\,\eta_3(\eta_{1/3}-\eta_{2/3}-1)\} \end{aligned} \tag{45}$$

For arbitrary n larger than 3, we can express $U(s)$ as follows:

$$U(s) = -\left(\frac{A_{n/1}}{2^{n-1}}\right) \sum_{j=1}^{2^{n-1}} \chi_j \sinh(\sqrt{s}\,\eta_n \omega_j) \tag{46}$$

because of

$$\mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_{n-1} \mathbf{E}_n = \left(\frac{A_{n/1}}{2^{n-1}}\right) \begin{pmatrix} E_{11}^{(n)} & E_{12}^{(n)} \\ E_{21}^{(n)} & E_{22}^{(n)} \end{pmatrix} \tag{47}$$

with

$$E_{11}^{(n)} = \sum_{j=1}^{2^{n-1}} \alpha_{j,n} \chi_j \cosh(\sqrt{s}\,\eta_n \omega_j) \tag{48}$$

$$E_{12}^{(n)} = -\sum_{j=1}^{2^{n-1}} \chi_j \sinh(\sqrt{s}\,\eta_n \omega_j) \tag{49}$$

$$E_{21}^{(n)} = -\sum_{j=1}^{2^{n-1}} \alpha_{j,n} \chi_j \sinh(\sqrt{s}\,\eta_n \omega_j) \tag{50}$$

$$E_{22}^{(n)} = \sum_{j=1}^{2^{n-1}} \chi_j \cosh(\sqrt{s}\,\eta_n \omega_j) \tag{51}$$

as shown in the Appendix. Here

$$\chi_j = \prod_{m=1}^{n-1} (A_{m/m+1} + \alpha_{j,m} \alpha_{j,m+1}) \tag{52}$$

$$\omega_j = \sum_{m=1}^{n} \alpha_{j,m} \eta_{m/n} \tag{53}$$

In Eqs. (48) to (53), all combinations of $\alpha_{j,m} = \pm 1$ are to be considered for $m \geqslant 2$ while $\alpha_{j,1} = 1$. It should be noted that the number of j which we must take into account depends on the number n of layers; $j = 2^{n-1}$. The temperature response at the rear surface can be expressed as

$$\Theta_n(0, s) = B_n = \left(\frac{2^{n-1}}{A_n}\right) \left[\frac{W(s)}{i\sqrt{s} \sum_{j=1}^{2^{n-1}} \chi_j \sin(-i\sqrt{s}\,\eta_n \omega_j)} \right] \tag{54}$$

Here use has been made of the relation that

$$\sinh z = i \sin(-iz), \qquad i = \sqrt{-1} \tag{55}$$

2.4. Inversion for the Laplace Transformation

The inversion of $\Theta_n(0, s)$ is not available in tables of the inverse Laplace transformation. Then the complex contour integration is required to obtain the inverse Laplace transformation:

$$\theta_n(0, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \Theta_n(0, s)\, ds \tag{56}$$

Here the poles for the transformation are the roots of the following characteristic equation including $\gamma = 0$:

$$U(\gamma) = \sum_{j=1}^{2^{n-1}} \chi_j \sin(\gamma \omega_j) = 0 \tag{57}$$

where

$$\gamma = -i \sqrt{s}\, \eta_n \tag{58}$$

It should be noted that the characteristic equation is independent of the heating method for the front surface of the sample material.

Introducing Cauchy's residue theorem for the integration, we obtain

$$\theta_n(0, t) = \sum_{k=0}^{\infty} R_k = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{p(s)}{q(s)} e^{st}\, ds \tag{59}$$

where R_k is a residue at a nonnegative kth pole. According to Heaviside's expansion theorem, residue at the pole of the first order is given by

$$R_k = \frac{p(s_k)}{q'(s_k)} e^{s_k t} \qquad \text{for} \quad s_k > 0 \tag{60}$$

Here a prime indicates d/ds.

3. TEMPERATURE RISE AT THE REAR SURFACE

3.1. Temperature Rise by Pulsewise Heating

When the front surface of the multilayered material is subjected to the pulsewise heating, $W(s)$ in Eq. (9) is expressed as

$$W(s) = \int_0^{\infty} Q\delta(0)\, e^{-st}\, dt = Q \tag{61}$$

where Q is the heat input per unit area and $\delta(t)$ the delta function. Functions of $p(s)$ and $q(s)$ in Eq. (59) are expressed as

$$p(s) = \left(\frac{2^{n-1}}{\Lambda_n \eta_n}\right) Q \tag{62}$$

and

$$q(s) = -\eta_n^{-2}\gamma \sum_{j=1}^{2^{n-1}} \chi_j \sin(\gamma\omega_j) \tag{63}$$

which yields

$$q'(s_k) = \left(\frac{1}{2}\right) \sum_{j=1}^{2^{n-1}} \omega_j \chi_j \cos(\gamma_k \omega_j) \qquad \text{for } \gamma_k \neq 0 \tag{64}$$

Here γ_k is the positive root of the characteristic equation of Eq. (57). Then from Eq. (60) the residue R_k is obtained as

$$R_k = \left(\frac{2^{n-1}Q}{\Lambda_n \eta_n}\right) \left[2 \frac{e^{-(\gamma_k/\eta_n)^2 t}}{\sum_{j=1}^{2^{n-1}} \omega_j \chi_j \cos(\gamma_k \omega_j)} \right] \tag{65}$$

For $\gamma_0 = 0$, that is, $s_0 = 0$, we have the residue as

$$R_0 = \left(\frac{2^{n-1}Q}{\Lambda_n \eta_n}\right) \left[\frac{1}{\sum_{j=1}^{2^{n-1}} \omega_j \chi_j} \right] \tag{66}$$

Then Eq. (59) becomes

$$\theta_n(0, t) = \left(\frac{2^{n-1}Q}{\Lambda_n \eta_n}\right) \left[\frac{1}{\sum_{j=1}^{2^{n-1}} \omega_j \chi_j} + 2 \sum_{k=1}^{\infty} \frac{e^{-(\gamma_k/\eta_n)^2 t}}{\sum_{j=1}^{2^{n-1}} \omega_j \chi_j \cos(\gamma_k \omega_j)} \right] \tag{67}$$

If we further normalize the temperature by the maximum temperature rise at the rear surface, Eq. (67) becomes

$$V = 1 + 2 \sum_{k=1}^{\infty} \frac{\left(\sum_{j=1}^{2^{n-1}} \omega_j \chi_j\right) e^{-(\gamma_k/\eta_n)^2 t}}{\sum_{j=1}^{2^{n-1}} \omega_j \chi_j \cos(\gamma_k \omega_j)} \tag{68}$$

It should be noted that the temperature rise expressed by Eq. (68) includes that for the single-layered material ($\omega_1 = \chi_1 = 1$) obtained by

Parker et al. [1] and those for two- and three-layered materials obtained by Lee [2] as special cases.

3.2. Temperature Rise by Stepwise Heating

When the front surface of the multilayered material is subjected to stepwise heating, $W(s)$ in Eq. (9) is expressed as

$$W(s)=\int_0^{\infty} Qe^{-st}\,dt=\frac{Q}{s} \tag{69}$$

where Q is the heat input per unit area. Functions of $p(s)$ and $q(s)$ in Eq. (59) are expressed as

$$p(s)=\left(\frac{2^{n-1}}{\Lambda_n \eta_n}\right)Q \tag{70}$$

and

$$q(s)=\eta_n^{-4}\gamma^3 \sum_{j=1}^{2^{n-1}} \chi_j \sin(\gamma\omega_j) \tag{71}$$

which yields

$$q'(s_k)=\left(\frac{\gamma_k^2}{2\eta_n^2}\right)\sum_{j=1}^{2^{n-1}} \omega_j\chi_j\cos(\gamma_k\omega_j) \qquad \text{for } \gamma_k\neq 0 \tag{72}$$

Here γ_k is the positive root of the characteristic equation of Eq. (57). Then from Eq. (60) the residue R_k is obtained as

$$R_k=\left(\frac{2^{n-1}Q\eta_n^2}{\Lambda_n\eta_n}\right)\left[-2\frac{e^{-(\gamma_k/\eta_n)^2 t}}{\gamma_k^2 \sum_{j=1}^{2^{n-1}} \omega_j\chi_j\cos(\gamma_k\omega_j)}\right] \tag{73}$$

For $\gamma_0=0$, that is, $s_0=0$, we have the residue as

$$R_0=\left(\frac{2^{n-1}Q}{\Lambda_n\eta_n}\right)\left[\frac{t}{\sum_{j=1}^{2^{n-1}}\omega_j\chi_j}\right]-\left(\frac{2^{n-1}Q\eta_n^2}{6\Lambda_n\eta_n}\right)\times\left[\frac{\sum_{j=1}^{2^{n-1}}\omega_j^3\chi_j}{\left(\sum_{j=1}^{2^{n-1}}\omega_j\chi_j\right)^2}\right], \tag{74}$$

because this pole is second order. Then Eq. (59) becomes

$$\theta_n(0, t) = \left[\frac{2^{n-1} Q \eta_n^2}{(\Lambda_n \eta_n) \sum_{j=1}^{2^{n-1}} \omega_j \chi_j} \right] \left[\left(\frac{t}{\eta_n^2} \right) - \left(\frac{\sum_{j=1}^{2^{n-1}} \omega_j^3 \chi_j}{6 \sum_{j=1}^{2^{n-1}} \omega_j \chi_j} \right) \right.$$

$$\left. - 2 \sum_{k=1}^{\infty} \frac{\left(\sum_{j=1}^{2^{n-1}} \omega_j \chi_j \right) e^{-(\gamma_k/\eta_n)^2 t}}{\gamma_k^2 \sum_{j=1}^{2^{n-1}} \omega_j \chi_j \cos(\gamma_k \omega_j)} \right] \tag{75}$$

From Eq. (75) we can obtain the temperature rise at the rear surface for three-layered material obtained by Araki and Natsui [3] as a special case.

3.3. Temperature Rise of the Functionally Gradient Material

Since we have obtained the temperature rise at the rear surface of the multilayered material as Eq. (68) for pulsewise heating and Eq. (75) for stepwise heating, let us apply them for the FGM in which thermophysical properties gradually change. If we consider the situation of $n \to \infty$, we can anticipate $\Lambda_{i/i+1} \approx 1$. This relation yields

$$\chi_1 = \prod_{m=1}^{\infty} (\Lambda_{m/m+1} + 1) \gg \chi_j \qquad (j > 1) \tag{76}$$

which means that all we must consider in the characteristic equation and the temperature rise is the first term for j; for ω's, it is enough to evaluate only ω_1, which is expressed as

$$\omega_1 \eta_\infty = \sum_{m=1}^{\infty} \eta_m = \int_{-L}^{0} \frac{dz}{\sqrt{a(z)}} \equiv S \tag{77}$$

where $a(z)$ is the profile function of the thermal diffusivity and L the thickness of the FGM. Since the characteristic equation of

$$\sin(\gamma_k \omega_1) = 0 \tag{78}$$

gives us the kth positive root as

$$\gamma_k \omega_1 = k\pi \qquad (k = 1, 2, \cdots) \tag{79}$$

the normalized temperature rise of Eq. (68) for the pulsewise heating is simply expressed as

$$V = 1 + 2 \sum_{k=1}^{\infty} \frac{e^{-[(k\pi)/S]^2 t}}{\cos(k\pi)} \tag{80}$$

It should be noted that the normalized temperature rise at the rear surface of the FGM coincides with that of the single-layered material.

For the stepwise heating, from Eq. (75), the ratio of the temperature rise defined as $V = \theta\ (0, 2t)/\theta\ (0, t)$ is expressed as

$$V = \frac{\left\{\frac{2t}{S^2} - \frac{1}{6} - 2 \sum_{k=1}^{\infty} \frac{e^{-2[(k\pi)/S]^2 t}}{(k\pi)^2 \cos(k\pi)}\right\}}{\left\{\frac{t}{S^2} - \frac{1}{6} - 2 \sum_{k=1}^{\infty} \frac{e^{-[(k\pi)/S]^2 t}}{(k\pi)^2 \cos(k\pi)}\right\}} \tag{81}$$

The characteristic equation is the same as Eq. (78).

3.4. Relation Between Apparent and Mean Thermal Diffusivities

Since we have successfully derived the normalized temperature rise of the FGM, let us investigate the relation between the thermal diffusivity obtained from temperature response and that related to the thermal resistance, taking pulsewise heating method as an example. In this investigation we should note that the thermal diffusivity obtained from temperature response regarding the FGM as a homogeneous material is apparent and that it is different from the mean thermal diffusivity which has physical meaning related to the thermal resistance. Although these points have already been pointed out by Araki et al. [4] for the two-layered material, we should also note that the temperature response has some information inside the sample material.

If we apply the conventional pulsewise heating method for Eq. (80), we formally obtain the apparent thermal diffusivity expressed as

$$a_e = \left(\frac{L}{\int_L^0 \frac{dz}{\sqrt{a(z)}}} \right)^2 \tag{82}$$

whereas the mean thermal diffusivity obtained from the thermal resistance is

$$a_m = \frac{1}{c_m \rho_m} \left(\frac{1}{\int_L^0 \frac{dz}{a(z)\,\rho(z)\,c(z)}} \right) \tag{83}$$

where $a(z)$ is the profile function of the thermal diffusivity inside the FGM and the mean values of the density and the specific heat capacity are, respectively,

$$\rho_m = \frac{1}{L} \int_{-L}^{0} \rho(z)\,dz, \qquad c_m = \frac{1}{\rho_m L} \int_{-L}^{0} \rho(z)\,c(z)\,dz \tag{84}$$

Therefore, we can say that the relation between the apparent and the mean thermal diffusivities is obtained from Eqs. (82) and (83) when the profile functions of the thermophysical properties are given. With this relation, the thermal conductivity, which is one of the important parameter for evaluation of the heat-shielding property, can be obtained with the pulsewise heating method.

4. TEMPERATURE RISE INSIDE THE MATERIAL

Next let us consider the temperature response inside the material. At a certain position z which locates in the ith layer, the temperature is expressed by the subsidary equation of Eq. (8). If we express it with the matrices defined in Section 2.2, we obtain

$$\Theta_i(z, s) = B_n (1 \quad 0)\, \mathbf{E}_i^* \mathbf{E}_{i+1} \cdots \mathbf{E}_{n-1} \mathbf{E}_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{85}$$

where

$$\mathbf{E}_i^* \equiv \begin{pmatrix} \cosh(\sqrt{s}\,\eta_n \eta_{i/n}^*) & -\sinh(\sqrt{s}\,\eta_n \eta_{i/n}^*) \\ -\sinh(\sqrt{s}\,\eta_n \eta_{i/n}^*) & \cosh(\sqrt{s}\,\eta_n \eta_{i/n}^*) \end{pmatrix} \tag{86}$$

and

$$\eta_i^* = (z_{i+1} - z)/\sqrt{a_i}, \qquad \eta_{i/n}^* = \eta_i^*/\eta_n \tag{87}$$

If we follow the procedure mentioned in the Appendix, it is shown that

$$\mathbf{E}_i^* \mathbf{E}_{i+1} \cdots \mathbf{E}_{n-1} \mathbf{E}_n = \left(\frac{A_{n/i}}{2^{n-i}} \right) \begin{pmatrix} E_{11}^{*(n-i+1)} & E_{12}^{*(n-i+1)} \\ E_{21}^{*(n-i+1)} & E_{22}^{*(n-i+1)} \end{pmatrix} \tag{88}$$

where

$$E_{11}^{*(n-i+1)} = \sum_{j=i}^{2^{n-i}} \alpha_{j,n-i+1}^{*} \chi_j^{*} \cosh(\sqrt{s}\,\eta_n \omega_j^{*}) \tag{89}$$

$$E_{12}^{*(n-i+1)} = -\sum_{j=i}^{2^{n-i}} \chi_j^{*} \sinh(\sqrt{s}\,\eta_n \omega_j^{*}) \tag{90}$$

$$E_{21}^{*(n-i+1)} = -\sum_{j=i}^{2^{n-i}} \alpha_{j,n-i+1}^{*} \chi_j^{*} \sinh(\sqrt{s}\,\eta_n \omega_j^{*}) \tag{91}$$

$$E_{22}^{*(n-i+1)} = \sum_{j=i}^{2^{n-i}} \chi_j^{*} \cosh(\sqrt{s}\,\eta_n \omega_j^{*}) \tag{92}$$

and

$$\chi_j^{*} = \prod_{m=i}^{n-1} (\Lambda_{m/m+1} + \alpha_{j,m}^{*}\,\alpha_{j,m+1}^{*}) \tag{93}$$

$$\omega_j^{*} = \eta_{i/n}^{*} + \sum_{m=i+1}^{n} \alpha_{j,m}^{*} \eta_{m/n} \tag{94}$$

In Eqs. (89) to (94), all combinations of $\alpha_{j,m}^{*} = \pm 1$ are to be considered for $m \geqslant i+1$, while $\alpha_{j,i}^{*} = 1$. Using Eq. (54), Eq. (85) can be expressed as follows:

$$\Theta_i(z, s) = \left(\frac{2^{i-1}}{\Lambda_i}\right) \left[\frac{W(s) \sum_{j=i}^{2^{n-i}} \chi_j^{*} \cos(-i\sqrt{s}\,\eta_n \omega_j^{*})}{i\sqrt{s} \sum_{j=1}^{2^{n-1}} \chi_j \sin(-i\sqrt{s}\,\eta_n \omega_j)} \right] \tag{95}$$

Through the inversion for the Laplace transformation (cf. Section 2.4), Eq. (95) yields the temperature rise for the pulsewise heating as

$$\theta_i(z, t) = \left(\frac{2^{i-1} Q}{\Lambda_i \eta_n}\right) \left(\frac{\sum_{j=i}^{2^{n-i}} \chi_j^{*}}{\sum_{j=1}^{2^{n-1}} \omega_j \chi_j} \right) V \tag{96}$$

where

$$V=1+2\sum_{k=1}^{\infty}\left[\frac{\sum_{j=i}^{2^{n-i}}\chi_j^*\cos(\gamma_k\omega_j^*)}{\sum_{j=i}^{2^{n-i}}\chi_j^*}\right]\left[\frac{\left(\sum_{j=1}^{2^{n-1}}\omega_j\chi_j\right)e^{-(\gamma_k/\eta_n)^2 t}}{\sum_{j=1}^{2^{n-1}}\omega_j\chi_j\cos(\gamma_k\omega_j)}\right] \tag{97}$$

is the normalized temperature by the maximum temperature rise at the rear surface.

For stepwise heating, Eq. (95) yields the temperature rise as follows:

$$\theta_i(z,t)=\left(\frac{2^{i-1}Q\eta_n^2}{\Lambda_i\eta_n}\right)\left(\frac{\sum_{j=i}^{2^{n-i}}\chi_j^*}{\sum_{j=1}^{2^{n-1}}\omega_j\chi_j}\right)\left\{\left(\frac{t}{\eta_n^2}\right)-\left(\frac{\sum_{j=1}^{2^{n-1}}\omega_j^3\chi_j}{6\sum_{j=1}^{2^{n-1}}\omega_j\chi_j}\right)\right.$$
$$\left.-2\sum_{k=1}^{\infty}\left[\frac{\sum_{j=i}^{2^{n-i}}\chi_j^*\cos(\gamma_k\omega_j^*)}{\sum_{j=i}^{2^{n-i}}\chi_j^*}\right]\left[\frac{\left(\sum_{j=1}^{2^{n-1}}\omega_j\chi_j\right)e^{-(\gamma_k/\eta_n)^2 t}}{\gamma_k^2\sum_{j=1}^{2^{n-1}}\omega_j\chi_j\cos(\gamma_k\omega_j)}\right]\right\}. \tag{98}$$

As for the FGM, the temperature rise for pulsewise heating is expressed as

$$V=1+2\sum_{k=1}^{\infty}\frac{\cos[k\pi S(z)/S]}{\cos(k\pi)}e^{-[(k\pi)/S]^2 t} \tag{99}$$

where

$$S(z)=\int_z^0\frac{dz}{\sqrt{\alpha(z)}},\qquad S(-L)\equiv S \tag{100}$$

For stepwise heating, it is expressed as

$$V=\frac{\left\{\frac{2t}{S^2}-\frac{1}{6}-2\sum_{k=1}^{\infty}\frac{\cos[k\pi S(z)/S]}{(k\pi)^2\cos(k\pi)}e^{-2[(k\pi)/S]^2 t}\right\}}{\left\{\frac{t}{S^2}-\frac{1}{6}-2\sum_{k=1}^{\infty}\frac{\cos[k\pi S(z)/S]}{(k\pi)^2\cos(k\pi)}e^{-[(k\pi)/S]^2 t}\right\}} \tag{101}$$

5. RESULTS

Numerial calculations have been performed in order to investigate the effects of the parameters on the temperature response. In the following, we restrict ourselves to the pulsewise heating method because nearly the same results are anticipated to be obtained for the stepwise heating method. Let us first study the temperature rise at the rear surface of the material, the front surface of which is subjected to the pulsewise heating. In the calculations, Eq. (68) is used for the multilayered material and Eq. (80) for FGM; the key parameters are the ratio of the heat-penetration coefficients, $\Lambda_{i/n}$, and the ratio of the thermal diffusion times, $\eta_{i/n}$. For simplicity, the following profiles of the thermophysical properties inside the multilayered material are assumed for the heat-penetration coefficient and the thermal diffusion time, respectively, as

$$\Lambda_{i/n}=1+\frac{n-i}{n-1}(\Lambda_{1/n}-1) \tag{102}$$

$$\eta_{i/n}=1+\frac{n-i}{n-1}(\eta_{1/n}-1) \tag{103}$$

with $\Lambda_{1/n}=10$ and $\eta_{1/n}=0.25$. These profiles become linear when the material is the FGM; therein the number n of layers inside the material is infinity.

Figure 2 shows the temperature rise at the rear surface of the material as a function of the Fourier number, with the number n of the layers taken as a parameter. The Fourier number in the abscissa is based on the apparent thermal diffusivity defined as

$$\mathrm{Fo}_e=\frac{a_e t}{L^2} \tag{104}$$

where a_e is the apparent thermal diffusivity obtained from the conventional application of the pulsewise heating method considering the sample material to be homogeneous. The use of this Fourier number enables us to normalize the time duration until which the temperature rise reaches half of the maximum temperature rise; for the FGM, that is, $n \to \infty$, the temperature response vs this Fourier number coincides with that for the single-layered material, as pointed out in Section 3.3. We see from Fig. 2 that the temperature rise depends on the number of layers inside the material. Here it should be noted that the apparent thermal diffusivity is not the thermophysical property as shown in Section 3.4.

Figure 2 also suggests that 11 layers are not enough to take the multilayered material as the FGM, as far as the temperature response is

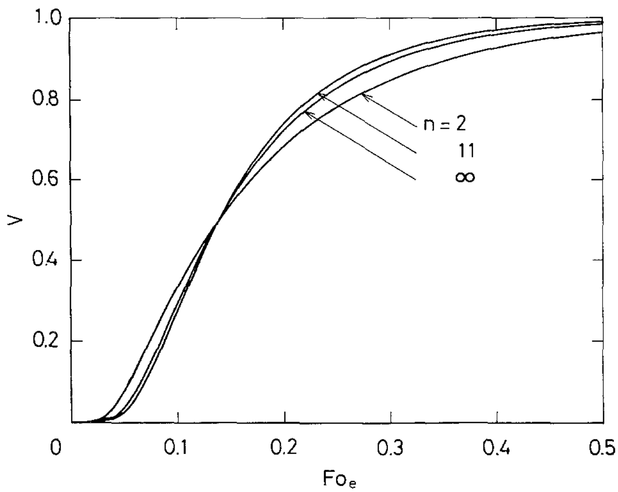

Fig. 2. Normalized temperature rise as a function of the Fourier number based on the apparent thermal diffusivity and thickness of the sample material, with the number of layers inside the material taken as a parameter; the ratio of the heat-penetration coefficients is $\Lambda_{1/n} = 10$; the ratio of the thermal diffusion times is $\eta_{1/n} = 0.25$.

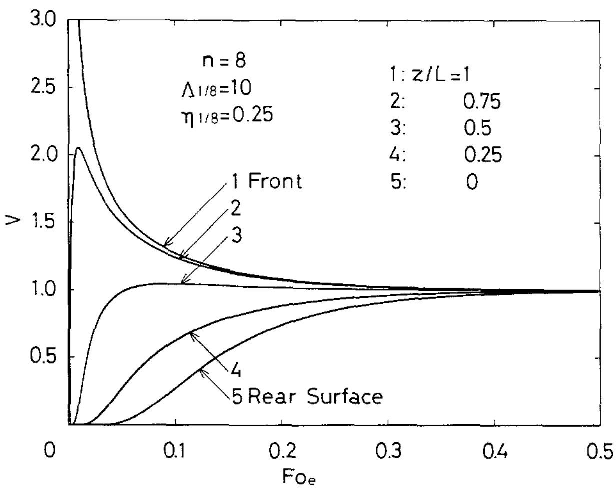

Fig. 3. Normalized temperature rise inside the eight-layered material as a function of the Fourier number based on the apparent thermal diffusivity and thickness of the sample material; the ratio of the heat-penetration coefficients is $\Lambda_{1/8} = 10$; the ratio of the thermal diffusion times is $\eta_{1/8} = 0.25$.

concerned. However, since the temperature rise has some information inside the sample material, the differences between temperature rises infer some possibilities to estimate the profile functions for the thermophysical properties; further studies are required.

Figure 3 shows the temperature rise inside the eight-layered material as a function of the Fourier number defined by Eq. (104). The profiles of the thermophysical properties inside the material are the same as those expressed by Eqs. (102) and (103), with $\Lambda_{1/8} = 10$ and $\eta_{1/8} = 0.25$. Although the temperature response of the FGM for the same condition is exactly the same as that for the single-layered material, the number of layers inside the sample material also influences the inside temperature response. This result also infers some possibilities to estimate the profile functions of the thermophysical properties from the deviation of the temperature rises. Again, further studies are required.

6. CONCLUDING REMARKS

In the present study, we have tried to derive the analytical solution of the temperature rise in the multilayered material which is subjected to transient heating, such as pulse- or stepwise heating. The general analytical solution for arbitrary multilayered material has successfully been obtained by solving the conventional heat diffusion equation with the appropriate initial and boundary conditions, with the use of the Laplace transformation. Since the present analytical solution includes that for up to three-layered material, it is expected to have general utility in the study of temperature response from which thermophysical properties are to be estimated. Furthermore, by considering the thickness of each layer to be infinitesimal, the present solution can be easily extended to that for the functionally gradient materials (FGM). Such materials are anticipated to improve the specific nature of the conventional materials and are intended to be used as electronic materials and/or materials resistant to wear, corrosion, and heat.

The present calculations emphasize that the temperature response varies depending on the number of layers in the sample material because of the change of the heat capacities in the sample material. Then as far as the temperature response is concerned, the present results suggest that great care must be taken when we take the multilayered material as the FGM; similarly, the FGM cannot simply be taken as the multilayered material. It has also been shown that the temperature response for the FGM coincides with that for the single-layered material if we use the Fourier number based on the apparent thermal diffusivity and the thickness of the sample material. This means that we can use the knowledge for the single-layered material in the literature when we evaluate the specific nature of the FGM from the thermophysical point of view. It should also be noted that the thermal diffusivity obtained from temperature response by taking the FGM as a homogeneous material is apparent and that it is different from the mean thermal diffusivity, which has physical meaning related to the thermal resistance.

Since we can obtain the temperature response of the arbitrary multilayered material, it may be the next subject to obtain or estimate the profile functions for the thermophysical properties, because the temperature response has some information inside the sample material. Further studies are then anticipated for the proper evaluation of the specific nature of the multilayered materials and/or the FGM.

APPENDIX

We prove here the general relation of Eq. (46) by mathematical induction. When $n=2$, from Eqs. (52) and (53) we obtain

$$\chi_1 = \Lambda_{1/2} + 1, \qquad \omega_1 = \eta_{1/2} + 1 \tag{A1}$$

$$\chi_2 = \Lambda_{1/2} - 1, \qquad \omega_2 = \eta_{1/2} - 1 \tag{A2}$$

with which we can derive Eq. (34) from Eq. (46). When $n=3$, we obtain

$$\chi_1 = (\Lambda_{1/2} + 1)(\Lambda_{2/3} + 1), \qquad \omega_1 = \eta_{1/3} + \eta_{2/3} + 1 \tag{A3}$$

$$\chi_2 = (\Lambda_{1/2} + 1)(\Lambda_{2/3} - 1), \qquad \omega_2 = \eta_{1/3} + \eta_{2/3} - 1 \tag{A4}$$

$$\chi_3 = (\Lambda_{1/2} - 1)(\Lambda_{2/3} - 1), \qquad \omega_3 = \eta_{1/3} - \eta_{2/3} + 1 \tag{A5}$$

$$\chi_4 = (\Lambda_{1/2} - 1)(\Lambda_{2/3} + 1), \qquad \omega_4 = \eta_{1/3} - \eta_{2/3} - 1 \tag{A6}$$

with which we can derive Eq. (40) from Eq. (46).

Next let us obtain the relation for $n=k+1$, assuming that the relation for $n=k$ is true. Equation (47) for $n=k+1$ can be expressed as

$$\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_k\mathbf{E}_{k+1} = \left(\frac{\Lambda_{k/1}}{2^{k-1}}\right)\begin{pmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ E_{21}^{(k)} & E_{22}^{(k)} \end{pmatrix}\Lambda_{k+1/k}$$
$$\times\begin{pmatrix} \cosh(\sqrt{s}\,\eta_{k+1}) & -\sinh(\sqrt{s}\,\eta_{k+1}) \\ -\Lambda_{k/k+1}\sinh(\sqrt{s}\,\eta_{k+1}) & \Lambda_{k/k+1}\cosh(\sqrt{s}\,\eta_{k+1}) \end{pmatrix} \tag{A7}$$

where

$$
\begin{aligned}
E_{11}^{(k)} &= \sum_{j=1}^{2^{k-1}} \alpha_{j,k}\chi_j \cosh(\sqrt{s}\,\eta_k\omega_j), \qquad & E_{12}^{(k)} &= -\sum_{j=1}^{2^{k-1}} \chi_j \sinh(\sqrt{s}\,\eta_k\omega_j) \\
E_{21}^{(k)} &= -\sum_{j=1}^{2^{k-1}} \alpha_{j,k}\chi_j \sinh(\sqrt{s}\,\eta_k\omega_j), \qquad & E_{22}^{(k)} &= \sum_{j=1}^{2^{k-1}} \chi_j \cosh(\sqrt{s}\,\eta_k\omega_j)
\end{aligned} \tag{A8}
$$

Then, Eq. (A7) becomes

$$
\mathbf{E}_1\mathbf{E}_2\cdots\mathbf{E}_k\mathbf{E}_{k+1} = \left(\frac{\Lambda_{k+1/1}}{2^k}\right)\begin{pmatrix} E_{11}^{(k+1)} & E_{12}^{(k+1)} \\ E_{21}^{(k+1)} & E_{22}^{(k+1)} \end{pmatrix} \tag{A9}
$$

where

$$
\begin{aligned}
E_{11}^{(k+1)} = \sum_{j=1}^{2^{k-1}} \Big[&\chi_j(\Lambda_{k/k+1}+\alpha_{j,k})\cosh\{\sqrt{s}\,(\eta_k\omega_j+\eta_{k+1})\} \\
&- \chi_j(\Lambda_{k/k+1}-\alpha_{j,k})\cosh\{\sqrt{s}\,(\eta_k\omega_j-\eta_{k+1})\}\Big]
\end{aligned} \tag{A10}
$$

$$
\begin{aligned}
E_{12}^{(k+1)} = -\sum_{j=1}^{2^{k-1}} \Big[&\chi_j(\Lambda_{k/k+1}+\alpha_{j,k})\sinh\{\sqrt{s}\,(\eta_k\omega_j+\eta_{k+1})\} \\
&+ \chi_j(\Lambda_{k/k+1}-\alpha_{j,k})\sinh\{\sqrt{s}\,(\eta_k\omega_j-\eta_{k+1})\}\Big]
\end{aligned} \tag{A11}
$$

$$
\begin{aligned}
E_{21}^{(k+1)} = -\sum_{j=1}^{2^{k-1}} \Big[&\chi_j(\Lambda_{k/k+1}+\alpha_{j,k})\sinh\{\sqrt{s}\,(\eta_k\omega_j+\eta_{k+1})\} \\
&- \chi_j(\Lambda_{k/k+1}-\alpha_{j,k})\sinh\{\sqrt{s}\,(\eta_k\omega_j-\eta_{k+1})\}\Big]
\end{aligned} \tag{A12}
$$

$$
\begin{aligned}
E_{22}^{(k+1)} = \sum_{j=1}^{2^{k-1}} \Big[&\chi_j(\Lambda_{k/k+1}+\alpha_{j,k})\cosh\{\sqrt{s}\,(\eta_k\omega_j+\eta_{k+1})\} \\
&+ \chi_j(\Lambda_{k/k+1}-\alpha_{j,k})\cosh\{\sqrt{s}\,(\eta_k\omega_j-\eta_{k+1})\}\Big]
\end{aligned} \tag{A.13}
$$

If we define

$$
(\Lambda_{k/k+1}\pm\alpha_{j,k}) = (\Lambda_{k/k+1}+\alpha_{j,k}\alpha_{j,k+1}) \tag{A14}
$$

$$
(\eta_k\omega_j\pm\eta_{k+1}) = (\eta_k\omega_j+\alpha_{j,k+1}\eta_{k+1}) \tag{A15}
$$

with $\alpha_{j,k+1} = \pm 1$, and redefine χ_j and ω_j as

$$
\chi_j = \prod_{m=1}^{k} (\Lambda_{m/m+1}+\alpha_{j,m}\alpha_{j,m+1}), \qquad \omega_j = \sum_{m=1}^{k+1} \alpha_{j,m}\eta_{m/k+1} \tag{A16}
$$

Equations (A10) to (A13) can be expressed as

$$
\begin{aligned}
E_{11}^{(k+1)} &= \sum_{j=1}^{2^k} \alpha_{j,k+1} \chi_j \cosh(\sqrt{s}\, \eta_{k+1} \omega_j), \\
E_{12}^{(k+1)} &= -\sum_{j=1}^{2^k} \chi_j \sinh(\sqrt{s}\, \eta_{k+1} \omega_j) \\
E_{21}^{(k+1)} &= -\sum_{j=1}^{2^k} \alpha_{j,k+1} \chi_j \sinh(\sqrt{s}\, \eta_{k+1} \omega_j), \\
E_{22}^{(k+1)} &= \sum_{j=1}^{2^k} \chi_j \cosh(\sqrt{s}\, \eta_{k+1} \omega_j)
\end{aligned}
\tag{A17}
$$

It should be noted that the number of j which we must take account of in the summation is

$$j = 2 \times 2^{k-1} = 2^{(k+1)-1} \tag{A18}$$

Equation (A17) for $n = k + 1$ coincides with Eq. (A8) for dependence on the number n. Therefore, the relation expressed in Eq. (47) is proven to be true by mathematical induction.

ACKNOWLEDGMENT

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NOMENCLATURE

A	Constant
a	Thermal diffusivity
B	Constant
$\mathbf{b}$	Vector defined in Eq. (18)
$\mathbf{C}$	Matrix defined in Eq. (18)
c	Specific heat capacity
$\mathbf{D}$	Matrix defined in Eq. (18)
E	Element of matrix $\mathbf{E}$
$\mathbf{E}$	Matrix $[=\mathbf{DC}^{-1}]$
Fo	Fourier number $[=at/L^2]$
L	Thickness of the sample material
l	Thickness of the layer
n	Number of layers in the sample material
p	Numerator of the image function for the temperature response
Q	Heat input per unit area
q	Denominator of the image function for the temperature response
R	Residue
S	Function defined in Eq. (100)

s Parameter in the Laplace transformation
t Time
U Laplace transform of the characteristic equation
V Temperature ratio
W Heat input function
$\mathbf{x}$ Vector defined in Eq. (31)
z Distance

Greek Symbols

α ± 1
γ Positive root of the characteristic equation
δ Delta function
η Thermal diffusion time $[=l/\sqrt{a}]$
Θ Laplace transform of the nondimensional temperature
θ Nondimensional temperature
Λ Heat-penetration coefficient $[=\lambda/\sqrt{a}]$
λ Thermal conductivity
ρ Density
χ Parameter defined in Eq. (52)
ω Parameter defined in Eq. (53)

Subscripts

e Apparent
i Value of the ith layer
i/j (Quantity of the ith layer) divided by (quantity of the jth layer)
m Mean

Superscripts

$'$ Differentiation
$*$ Inside the layer

REFERENCES

1. W. J. Parker, R. J. Jenkins, C. P. Butler, and G. I. Abbott, *J. Appl. Phys.* **32**:1679 (1961).
2. T. Y. R. Lee, *Thermal Diffusivity of Dispersed and Layered Composites*, Ph.D. thesis (Purdue University, Lafayette, IN, 1977).
3. N. Araki and K. Natsui, *Trans. Japan Soc. Mech. Engr. (Ser. B)* **49**:1048 (1983) (in Japanese). [Translated into English in *Heat Transfer Japan. Res.* **14**:36 (1985).]
4. N. Araki, A. Makino, and J. Mihara, *Int. J. Thermophys.* **13**:331 (1992).